

QUANTUM GEOMETRY AND THE PLANCK SCALE

S. Majid¹

Department of Mathematics, Harvard University
Science Center, Cambridge MA 02138, USA²

+

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW

December 1996

We consider some general aspects of the new noncommutative or quantum geometry coming out of the theory of quantum groups, in connection with Planck scale physics. A generalisation of Fourier or wave-particle duality on curved spaces emerges. Another feature is the need for particles with fractional or braid statistics. The conformal group also has a special role.

1 Introduction

In this contribution I would like to discuss some of the conceptual and physical issues surrounding the approach to noncommutative geometry coming out of quantum groups and braided groups, in somewhat greater depth than I had time for in my lecture in Goslar. The discussion is intended to be intelligible to non-specialists and may hopefully serve as an invitation to the field. Technical material including recent results in quantum and braided geometry may be found in my contribution to the companion ‘Quantum Groups’ volume of these Proceedings. Basic material can also be found in my textbook on quantum groups[1].

The need for some kind of quantum geometry has been clear enough since the birth of quantum mechanics itself: how to extend ideas of gauge theory, curvature and non-Euclidean geometry to the the situation where coordinates are noncommuting operators. If one ever wants to unify quantum mechanics and gravity into a single exact theory then this is probably a prerequisite. We begin, in Section 2, by postulating some fundamental features which any satisfactory such quantum geometry should have. One of them is an extension of wave-particle duality to curved space[2][3].

In Section 3 we consider quantum groups as examples of quantum geometry. Actually, this is not at all the context in which the more famous q -deformed enveloping algebras $U_q(g)$

¹Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge

²During the calendar years 1995 + 1996

arose (which is that of ‘generalised symmetries’ of exactly solvable lattice models; there q is an anisotropy parameter and not directly related to Planck’s constant). However, at about the same time as these $U_q(g)$ were being introduced in the mid 1980’s, another completely different class of quantum groups $\mathbb{C}(G_1) \bowtie U(g_0)$ was being introduced (by the author) in another context, which was precisely the context of Planck scale physics and an algebraic approach to quantum-gravity. These *bicrossproduct* quantum groups really do arise as quantum algebras of observables[3]. Here $G_0 G_1$ is a Lie group factorisation and g_0 is the Lie algebra of G_0 . More recently, gauge theory etc. has been developed for quantum geometry[4][5].

As soon as spaces become noncommutative or ‘quantum’, their symmetries naturally become generalised as quantum groups too. Here the quantum groups $U_q(g)$ play a more central role. It turns out as a new feature of quantum geometry that anything on which a quantum group acts acquires braid statistics. In other words. not only are algebras noncommutative but tensor products become noncommutative too. This means that at the Planck scale one should expect not only bosons and fermions but complicated braid statistics as well. This gives a systematic ‘braided approach’[6] to quantizing everything uniformly, and is the topic of Section 4.

Obviously everything we may want to say about the Planck scale here will be speculative. However, mathematics can tell us that certain assumptions will force us to certain conclusions on the mathematical structure, without yet knowing realistic models. Moreover, quantum geometry is probably necessary not only at the Planck scale or in quantum cosmology, but also in the resolution of those paradoxes in quantum mechanics which are characterised by a conflict between the macroscopic geometry of measuring apparati and quantum mechanical evolution; it should provide the right language to correctly formulate such questions.

Note also that most discussions of Planck scale physics, including string theory, work within an underlying classical geometry (e.g. inside a path integral.) This is practical but not really justified: classical geometry should emerge from a deeper quantum world and not vice-versa. Even Professor Fredenhagen in his talk *assumed exact classical Poincaré symmetry* without any real justification, except that this is a necessary assumption to be able apply the known methods of algebraic quantum field theory. This is still ‘looking for the key under the streetlight’; by contrast the quantum geometry programme advocated here seeks primarily to understand first what mathematically natural quantum geometry is out there, before making predictions. The success of General Relativity can be attributed in part to the fact that Einstein already

had a fairly complete mathematical theory of Riemannian geometry to bring to bear. For example, modified uncertainty relations based on modified commutation relations $[x, p]$ are only as meaningful as the justification that a particular operator x be called position and a particular operator p be called momentum, which should generally come from a quantum geometrical picture.

2 Inventing quantum geometry

Riemannian geometry as we know it arose in two stages. First, one can work extrinsically with surfaces embedded in Euclidean space, or submanifolds $M \subset \mathbb{R}^n$. This is a Gaussian approach. Riemann realised, however, that one needs also a more intrinsic notion of geometry determined by structure on M itself, particularly since actual space or (after Einstein) spacetime is what we directly experience rather than any embedding space. This leads to our modern form of non-Euclidean or Riemannian geometry. The situation with quantum mechanics can be viewed analogously. The role of \mathbb{R}^n is played in a certain sense by $B(\mathcal{H})$, the operators on a Hilbert space. However, the real observables in the system are usually some $*$ -subalgebra $A \subset B(\mathcal{H})$. We broaden the problem a little and consider the intrinsic structure of quite general algebras A (not only normed $*$ -algebras over \mathbb{C}). In my opinion we are in a similar situation to Riemann: how to develop a language to describe the intrinsic geometric structure on an algebra. Note that for any manifold we can consider the algebra of smooth functions on it and formulate our usual geometrical notions in those terms; the key difference in quantum geometry is that we do not want to be limited to commutative algebras.

Also, let us say from the start that the real physical motivation for quantum geometry applies directly only to phase spaces, which lose their points on quantisation due to the uncertainty principle; the coordinates of phase space no longer fully commute. However, a generalised framework for phases spaces probably entails developing a generalised framework for manifolds in general. Apart from this zeroth assumption, we postulate also:

1. **Richness** – at least as much ‘flabbiness’ in the variety of examples and structures (gauge fields, curvature etc.) as classically.
2. **Quantization without classical assumptions** – classical geometry should emerge as a possible limit, not be built in from the start by assuming a Poisson manifold.

3. **Uniformity of quantisation** – the whole geometrical ‘zoo’ should be quantised together, coherently, not one object at a time.
4. **Duality between geometry and quantum mechanics** – wave particle duality should be maintained in some form.

Axiom 1 here is not as empty as it may seem. There are many noncommutative algebras but most of them will be too wild to fit recognisably into a geometric picture. In the early 1980’s most papers on noncommutative geometry focussed on the noncommutative torus as the main noncommutative example. Quantum geometry today contains q -planes, q -spheres, q -groups G_q (the coordinate rings of $U_q(g)$), q -monopoles, q -Lie algebras, q -vector fields etc.

Axiom 2 involves a slight abuse of notation – what is quantisation if not a process starting with a Poisson manifold? The idea is that we need deeper more intrinsically algebraic notions for the construction of quantum algebras, from which Poisson manifolds can sometimes (though not necessarily) be obtained by ‘classicalisation’ with respect to a choice of generators (cf. Lie algebra contraction). Quantum geometry at present supplies two; one is the idea of factorisation and the other is the idea of R-matrix or quasitriangular structures.

Axiom 3 is a more novel issue, usually overlooked even in a Poisson geometric setting: in classical geometry we demand commutativity *uniformly* for all coordinate algebras. When we relax it for one geometrical object (e.g. position space), should we not relax it for another (e.g. momentum space)? And there are many different ‘directions’ in which one can relax commutativity (e.g. many choices of Poisson structure or R-matrix) for each object and we need to choose these consistently. Our quantum spheres have to be consistent with our quantum planes etc. Section 4 explains how this can be achieved by means of braid statistics (the braided approach to q -deformation).

Axiom 4 probably needs the most explanation. A modern way to think about wave-particle duality is in terms of Fourier theory. The dual group to position space \mathbb{R} (the group of irreducible representations) is again \mathbb{R} , the momentum space. On the one hand, points x are fundamental ‘particles’ while on the other hand waves or points p in momentum space are fundamental ‘particles’; the two points of view are related by Fourier transform. When position space is curved, e.g. a simple Lie group G , then the irreducible representations \hat{G} do not form a group. However, non-Abelian Fourier theory is still possible with the right mathematical generalisation. Note that G itself is ‘geometrical’ while the ‘points’ of \hat{G} are more like quantum states, so this

generalised Fourier theory is an example of a *quantum-geometry transformation*. We would like a similar elucidation for some class of ‘group’ objects and their duals in any quantum geometry. Ideally, we might hope for a stronger form of Axiom 4, which we call the *principle of representation-theoretic self-duality*: quantum geometry should be general enough that the duals of ‘group’ objects are again ‘group’ objects. This is the case for the kind of quantum geometry coming of quantum groups and braided groups in Sections 3,4. Moreover, some groups, like \mathbb{R}^n , are self-dual. The self-dual ‘groups’ in quantum geometry likewise occupy a special place as the simplest geometrical objects.

Actually, we have argued[2] that the bifurcation into ‘geometrical’ and ‘quantum’ ideas (dual to each other and related by a generalised Fourier or quantum geometry transformation) has its origin in our concept of physical reality itself (the dual nature of measurement and object being measured), and is not really tied to groups. More general manifolds with ‘geometrical’ structure have more complex dual notions of ‘representation’, etc. and self-duality will pick out geometries occupying a special place. We have postulated this self-duality constraint for the quantum geometry of phase space as the philosophical origin of something like Einstein’s equation[2].

3 Elements of quantum geometry

The present approach to quantum geometry is motivated particularly from the duality Axiom 4, which leads one to formulate group objects as Hopf algebras. A Hopf algebra is a unital algebra H and algebra homomorphisms $\Delta : H \rightarrow H \otimes H$ (coproduct), $\epsilon : H \rightarrow \mathbb{C}$ (counit) forming a counital coalgebra. The axioms of a counital coalgebra are just those of a unital algebra with all arrows reversed (think of the unit element as a map $\mathbb{C} \rightarrow H$). So H^* is also a unital algebra by dualising Δ, ϵ . There is also a kind of ‘linearised inversion’ $S : H \rightarrow H$ called the antipode.

In a suitable setting, if H is a Hopf algebra then essentially H^* is another Hopf algebra by dualisation. The product of one determines the coproduct of the other and vice-versa. So Axiom 4 is satisfied in the strong form. Moreover, if G is a group then its coordinate ring $\mathbb{C}(G)$ is a Hopf algebra, at least in nice cases. In the finite case we mean all functions f on G with pointwise product and $\Delta f = f(\cdot)$, where \cdot is the group product. The blanks on the right hand side indicate a function of two variables, i.e. an element of $\mathbb{C}(G) \otimes \mathbb{C}(G)$. So Hopf algebras generalise the notion of usual groups. Dually paired to $\mathbb{C}(G)$ is the group algebra $\mathbb{C}G$ (the linear

extension of G with $\Delta g = g \otimes g, \epsilon g = 1$) in the finite case or enveloping algebra $U(g)$ in the Lie group case with Lie algebra g . Fourier theory is a linear isomorphism $H \rightarrow H^*$ or $\mathbb{C}(G) \rightarrow \mathbb{C}G$ in the finite case. When G is Abelian, we have $\mathbb{C}G = \mathbb{C}(\hat{G})$, the coordinate ring of the dual group. But when G is non-Abelian, $\mathbb{C}G$ is not commutative, so then \hat{G} only exists as a quantum geometry with, by definition, coordinate ring $\mathbb{C}G$. One should consider any noncommutative Hopf algebra as, by definition, a quantum group.

3.1 Toy models of quantum-gravity

Here we will see how the structural considerations in Section 2 can force one to concrete Planck scale dynamics. For our discussion, all quantum geometries that we consider will be ‘group’ objects, i.e. Hopf algebras, but the ideas could ultimately be applied more generally.

Suppose that we fix the position Hopf algebra H_1 and momentum Hopf algebra H_0 . Instead of Poisson brackets or other guesswork about the quantum phase space (i.e. instead of guessing position-momentum commutation relations) let us proceed structurally and consider all possible extensions

$$H_1 \rightarrow E \rightarrow H_0. \quad (1)$$

This means a Hopf algebra E and Hopf algebra maps as shown, obeying certain conditions[1]. Then theory tells us that E will be a cocycle bicrossproduct $E = H_1 \blacktriangleright\blacktriangleleft H_0$. The simplest case is with trivial cocycles, in some cohomological sense ‘close’ to the tensor product $H_1 \otimes H_0$. Then the theorem is that E is a cross product by an action of H_0 on H_1 and a cross coproduct by a coaction of H_1 on H_0 .

For example, let us consider *all possible* extensions

$$\mathbb{C}[x] \rightarrow E \rightarrow \mathbb{C}[p] \quad (2)$$

of 1-dimensional position and momentum spaces $\mathbb{C}[x]$ and $\mathbb{C}[p]$. These are classical groups and hence Hopf algebras, with $\Delta x = x \otimes 1 + 1 \otimes x$ and $\Delta p = p \otimes 1 + 1 \otimes p$. We consider all possible E , i.e. do not build any relations in by hand, and we consider the Hamiltonian to be fixed in the form $p^2/2m$. Note that this approach is more intrinsic (in the spirit of Einstein’s equivalence principle) than keeping the commutation relations in some canonical form but varying the Hamiltonian.

Proposition 3.1 [3] *The possible cocycle-free extensions (2) are described by two parameters*

\hbar, G and take the form $E_{\hbar, G} = \mathbb{C}[x] \blacktriangleright \mathbb{C}[p]$ with cross relations and coproduct

$$[x, p] = i\hbar(1 - e^{-\frac{x}{MG}}), \quad \Delta p = p \otimes e^{-\frac{x}{MG}} + 1 \otimes p,$$

where M is a convenient fixed constant of mass dimension.

The free-fall Hamiltonian $p^2/2m$ gives $\dot{x} = v_\infty(1 - e^{-\frac{x}{MG}})$ to lowest order in \hbar , which can be compared with infalling radial coordinates $\dot{x} = -(1 - (1 + \frac{x}{2MG})^{-1})$ near a black hole of mass M . So the parameter G plays a role in this simple model similar to the gravitational coupling constant. Also, the particle moves more and more slowly as it approaches the origin and takes an infinite time to reach it. Yet when G is small, the commutation relations are $[x, p] = i\hbar$ at least for states where one can say that $x > 0$, i.e. away from the origin. Further analysis of this model gives the effective scales for these gravitational and quantum limits as $mM \gg m_P^2$ and $mM \ll m_P^2$, where $m_P = \sqrt{\hbar/G}$ is the Planck mass [3][1].

The same ideas work when the position and momenta are curved. Let G_1 be a Lie group and g_0 a Lie algebra with group G_0 . One can consider cocycle-free extensions

$$\mathbb{C}(G_1) \rightarrow E \rightarrow U(g_0). \quad (3)$$

The possible extensions turn out[7] to correspond essentially to solutions of the factorisation problem: Lie groups factorising into $G_0 G_1$. For example, the complexification of any compact real form G_0 of a simple Lie group factorises as $G_0 G_1$ for a certain solvable G_1 , and gives a corresponding E . The natural Hamiltonian is the quadratic the Casimir of g_0 and induces quantum dynamics on G_1 as position space.

For example, the quantum group $E = \mathbb{C}(\mathbb{R}^2 \blacktriangleright \mathbb{R}) \blacktriangleright U(so_3)$ corresponds to the Iwasawa factorisation of $SL(2, \mathbb{C})$. One can insert two free parameters \hbar, G as well. Then E is generated by the coordinates x_i of $\mathbb{R}^2 \blacktriangleright \mathbb{R}$ and e_i of su_2 , with[1]

$$\begin{aligned} [e_i, e_j] &= i\hbar \epsilon_{ijk} e_k, \quad [e_i, x_j] = i\hbar \epsilon_{ijk} x_k - \frac{i\hbar}{2MG} \epsilon_{ij3} x \cdot x (1 + \frac{x_3}{MG})^{-1} \\ [x_i, x_j] &= 0, \quad \Delta x_i = x_i \otimes 1 + (1 + \frac{x_3}{MG}) \otimes x_i \\ \Delta e_i &= e_i \otimes (1 + \frac{x_3}{MG})^{-1} + \frac{1}{MG} e_3 \otimes x_i (1 + \frac{x_3}{MG})^{-1} + 1 \otimes e_i. \end{aligned} \quad (4)$$

By thinking of the x_i as momenta p_i (wave-particle duality again), one can also consider this E as some kind of deformation of $U(\mathbb{R}^3 \blacktriangleright so_3)$, i.e. of the Poincaré enveloping algebra in 3 dimensions. So on the one hand, E is a quantisation of particles on orbits in \mathbb{R}^3 exhibiting

singular dynamics, and on the other it is a deformation of a symmetry algebra. The Minkowski spacetime version of this model is very similar and of independent interest [8].

Moreover, E^* is another quantum phase space. It solves the extension problem

$$H_0^* \rightarrow E^* \rightarrow H_1^*. \quad (5)$$

When position space is flat as in (2) then essentially $\mathbb{C}[x]^* = \mathbb{C}[p]$ (wave particle duality) and E^* also solves (2). Hence it is of the same form as in Proposition 3.1, i.e. these E are self-dual quantum groups. In the curved position space case, the dual quantum group is $E^* = U(g_1) \bowtie \mathbb{C}(G_0)$. For example, the dual Hopf algebra to (4) describes quantum particles in $SU_2 = S^3$ moving on orbits under $\mathbb{R}^2 \rtimes \mathbb{R}$. The explicit orbits and flows in these models are obtained by solving nonlinear equations. Also, one can classicalise and obtain Poisson manifolds of which these quantum groups are quantisation, although they would not be determined uniquely as such; see [1].

This demonstrates our algebraic approach to Planck scale physics. It is one of the historical origins of noncommutative (and noncocommutative) Hopf algebras or quantum groups. Recent work on bicrossproducts is in [9].

3.2 Quasitriangular structures

It would be remiss not to mention the more famous Drinfeld-Jimbo quantum groups $U_q(g)$ and their duals G_q . They have little, so far, to do with Planck scale physics (as far as I know), arising independently in quite a different physical context. They can, however, be classicalised and hence viewed (if we want) as quantisations of a certain Drinfeld-Sklyanin Poisson bracket on the Lie group of g . At this level, there are connections with the factorisation problem above[7]. Also, they again demonstrate our Axiom 2 that quantum geometry has its own intrinsic structure. The intrinsic structure of $U_q(g)$ is that of a *quasitriangular* Hopf algebra[10]. It is a Hopf algebra H equipped with a so-called (by physicists) universal R-matrix $\mathcal{R} \in H \otimes H$. Its image in any matrix representation obeys the Yang-Baxter or braid relations. Such generalised symmetry algebras are relevant to the next section.

The intrinsic structure of G_q is therefore that of a *dual quasitriangular* Hopf algebra. This is a Hopf algebra H equipped with a skew bicharacter $\mathcal{R} : H \otimes H \rightarrow \mathbb{C}$ obeying

$$g_{(1)} h_{(1)} \mathcal{R}(h_{(2)}, g_{(2)}) = \mathcal{R}(h_{(1)}, g_{(1)}) h_{(2)} g_{(2)} \quad (6)$$

for all $h, g \in H$, where $\Delta h = h_{(1)} \otimes h_{(2)}$ is our notation for the coproduct with output in $H \otimes H$ (summation omitted). When \mathcal{R} , G_q are expanded in \hbar with $q = e^{\hbar/2}$, one obtains a Poisson bracket. But the quantum world is richer. For example, there are discrete quantum groups possessing such \mathcal{R} . Thus the axioms for H, \mathcal{R} carve out a class of quantum groups defined intrinsically and ‘close’ to being commutative in the sense (6) rather than in the conventional sense of quantisation of a Poisson bracket.

4 Elements of braided geometry

In this section we explain another approach to quantum geometry, which has so far been applied mostly in flat space (rather than having direct contact with the Planck scale), but which has the merit of solving the uniformity Axiom 3. Ultimately, we would like to see it combined with the ideas in the preceding section. This *braided geometry* involves a new kind of mathematics in which information ‘flows’ along braids and tangles much as it flows along the wiring in a computer, except that under- and over-crossings of wires are now nontrivial *braiding operators* Ψ . In usual mathematics and computer science one wires outputs of operations into inputs of other operations without caring about such crossings, i.e. usual mathematics is two-dimensional. By contrast, braided calculations, braided Feynman diagrams etc. truly exist in a three-dimensional space where calculations take place. Mathematically, we make use of the theory of braided categories [11]. The introduction of algebras, group theory and geometry in braided categories is due to the author, e.g.[12][6].

The idea is that in quantum physics there is another kind of noncommutativity, namely anticommutativity due to fermionic statistics. This is a noncommutativity of \otimes itself. Thus, when independent fermionic systems must be exchanged during a manipulation, one uses supertransposition $\Psi(b \otimes c) = (-1)^{|b||c|} c \otimes b$, where $| \cdot |$ is the degree 0, 1. For example, the supertensor product $B \otimes C$ of two superalgebras involves Ψ , with the result that $cb \equiv (1 \otimes c)(b \otimes 1) = \Psi(c \otimes b) = (-1)^{|c||b|} bc$ in $B \otimes C$. The idea of braided geometry is that Ψ , and hence the cross relations of $B \otimes C$, can be much more general than this simple ± 1 form. When Ψ^2 is not always the identity, one says that the system has *braid statistics*. Thus,

- quantum geometry: \otimes usual commutative (bosonic) one, coordinate algebras noncommutative.

- braided geometry: \otimes non commutative (braid statistics), coordinate algebras may as well be ‘commutative’ in suitably modified sense.

Just as quantum mechanics was created with the realisation that many construction do not require commutativity of coordinates, braided geometry is created by a second and equally deep realisation: *many constructions do not require commutativity of the notion of independence*. In particular, we can take in place of -1 a dimensionless parameter q or, more generally, an operator Ψ depending on one or more parameters q . Moreover, *classical braided geometry \Rightarrow quantum geometry* in a q -deformed sense because ‘braided commutative’ generally means noncommutative with respect to the usual \otimes . Moreover, specifying the braid statistics specifies such things coherently between every object and every other object. Quantum groups still play a role:

Proposition 4.1 *cf.[10] All objects B, C on which a quantum group like $U_q(g)$ (a quasitriangular Hopf algebra) acts acquire braid statistics $\Psi(b \otimes c) = \mathcal{R}_i.c \otimes \mathcal{R}^i.b$, where $\mathcal{R} = \mathcal{R}^i \otimes \mathcal{R}_i$ is the universal R -matrix or quasitriangular structure.*

An example is the quantum-braided plane \mathbb{C}_q^2 generated by a vector of coordinates $\mathbf{x} = (x, y)$ obeying $yx = qxy$. It has braiding and braided-coproduct[13]

$$\begin{aligned} \Psi(x \otimes x) &= q^2 x \otimes x, & \Psi(y \otimes y) &= q^2 y \otimes y, & \Psi(x \otimes y) &= qy \otimes x \\ \Psi(y \otimes x) &= qx \otimes y + (q^2 - 1)y \otimes x, & \Delta \mathbf{x} &= \mathbf{x} \otimes 1 + 1 \otimes \mathbf{x}. \end{aligned} \tag{7}$$

There are braided-plane structures for q -Euclidean and q -Minkowski spaces. They are isomorphic to their duals (q -wave-particle duality). There are also braided matrices $B(R)$ generated by $\mathbf{u} = (u^i_j)$ with relations $R_{21}\mathbf{u}_1 R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1 R$ and[12]

$$\Delta \mathbf{u} = \mathbf{u} \otimes \mathbf{u}, \quad \Psi(\mathbf{u}_1 \otimes R \mathbf{u}_2) = R \mathbf{u}_2 R^{-1} \otimes \mathbf{u}_1 R \tag{8}$$

in a compact notation, for any biinvertible matrix R obeying the Yang-Baxter equations. Their quotients by q -determinant and other relations give braided versions BG_q of the coordinate rings of simple Lie groups. They are dual to braided versions $BU_q(g)$ of the enveloping algebras. Here is a remarkable selfduality phenomenon:

Proposition 4.2 [6][14] *When $q \neq 1$ one has essentially $BG_q \cong BU_q(g)$.*

So in the $q \neq 1$ world there is essentially only one q -deformed object for each simple Lie algebra g , which has two limits as $q \rightarrow 1$. On the left hand side it becomes $\mathbb{C}(G)$ the commutative

coordinate ring. On the right hand side it becomes the enveloping algebra $U(g)$. Thus these two features of classical mathematics, conceptually dual to each other, are different scaling limits of one object. In a similar way, one finds that q -Minkowski space as a 2×2 braided matrix is isomorphic to the braided enveloping algebra of a braided Lie algebra version of $su_2 \oplus u(1)$ [14]. This is wave-particle duality in a strong form, and is only possible when $q \neq 1$.

The q -Poincaré and q -conformal groups are also obtained from braided geometry. With q -Minkowski space as an additive braided group (like the quantum-braided plane above) one has a braided adjoint action of the braided-coordinates on themselves. This is not possible when $q = 1$ since the adjoint action is then trivial. However, when $q \neq 1$ it generates the action of q -special conformal transformations[15]. The remnant of this as $q \rightarrow 1$ is

$$I \circ \frac{\partial}{\partial x_i} \circ I = \lim_{q \rightarrow 1} \frac{\text{Ad}_{x_i}}{q - q^{-1}} \quad (9)$$

where I is conformal inversion. This is a completely new group-theoretical picture of conformal transformations as adjoint action, only possible when $q \neq 1$.

The above approach to q -deformation has been developed over 50–60 papers by the author and collaborators since 1989. It provides the correct meaning of q as ‘braid statistics’ (rather than directly related to \hbar) and a systematic solution to the problem of q -deforming everything. Moreover, we see that our familiar $q = 1$ world is merely a special limit of a deeper and more natural $q \neq 1$ geometry.

Acknowledgments

I would like to thank the organizers, H.-D. Doebner, P. Kramer, W. Scherer, V. Dobrev and others for an outstanding conference.

References

- [1] S. Majid, *Foundations of quantum group theory*, Cambridge Univ. Press, 1995.
- [2] S. Majid, “Principle of representation-theoretic self-duality”, *Phys. Essays* **4**, 395–405 (1991).
- [3] S. Majid, “Hopf algebras for physics at the Planck scale”, *J. Classical and Quantum Gravity* **5**, 1587–1606 (1988).

- [4] T. Brzeziński and S. Majid, “Quantum group gauge theory on quantum spaces”, *Commun. Math. Phys.* **157**, 591–638 (1993); erratum **167**, 235 (1995).
- [5] S. Majid, “Diagrammatics of braided group gauge theory”, preprint, 1996.
- [6] S. Majid, “Beyond supersymmetry and quantum symmetry”, in *Quantum groups, integrable statistical models and knot theory* (Eds. M-L. Ge and H.J. de Vega), p. 231–282, World Sci., 1993.
- [7] S. Majid, “Physics for algebraists: non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction”, *J. Algebra* **130**, 17–64 (1990).
- [8] S. Majid and H. Ruegg, “Bicrossproduct structure of the κ -Poincaré group and non-commutative geometry”, *Phys. Lett. B* **334**, 348–354 (1994).
- [9] E. Beggs and S. Majid, “Quasitriangular and differential structures on bicrossproduct Hopf algebras”, Preprint, 1996.
- [10] V.G. Drinfeld, “Quantum groups”, in *Proceedings of the ICM* (Ed. A. Gleason), p. 798–820, AMS, Rhode Island, 1987.
- [11] A. Joyal and R. Street, “Braided monoidal categories”, Macquarie preprint, 1986.
- [12] S. Majid, “Examples of braided groups and braided matrices”, *J. Math. Phys.* **32**, 3246–3253 (1991).
- [13] S. Majid, “Braided momentum in the q -Poincaré group”, *J. Math. Phys.* **34**, 2045–2058 (1993).
- [14] S. Majid, “Quantum and braided Lie algebras”, *J. Geom. Phys.* **13**, 307–356 (1994).
- [15] S. Majid, “Braided geometry of the conformal algebra”, *J. Math. Phys.* (1996).